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Quantum teleportation of single-mode thermal state of light field

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Abstract
The quantum teleportation of a single-mode thermal state of a light field is discussed using the projection synthesis technique in the Schrödinger picture. The statistical distance (SD) between the teleported input and output states is introduced to evaluate the efficiency of single-mode thermal state teleportation. For a given system and available entanglement, both the SD and classical boundary depend on the mean photon numbers of the teleported thermal light field. The dependences of the SD on the normalized classical gain, EPR (Einstein–Podolsky–Rosen) entanglement and mean photon numbers of the thermal field are calculated.

Keywords: quantum teleportation, EPR (Einstein–Podolsky–Rosen) entanglement

1. Introduction

Quantum teleportation has attracted considerable interest in recent years as important progress in quantum information science. It was first proposed by Bennett *et al* in the context of spin 1/2 particles [1] and was realized in experiments for photon polarization states [2, 3]. Recently the teleportation of continuous variables was proposed [4–10], then the teleportation of an optical coherent state was experimentally demonstrated [11] by exploiting a squeezed vacuum state of light to provide EPR (Einstein–Podolsky–Rosen) entanglement. The teleportation of non-local entanglement, i.e. entanglement swapping [12], which can be used for establishing non-local correlations over a large distance, has also been investigated [13–15]. The experimental demonstration of entanglement swapping was accomplished in discrete-variable systems [16]. The continuous-variable entanglement teleportation was suggested for both polarization-entangled states and entangled squeezed states using a squeezed vacuum state as the non-ideal EPR source [17–20], and it was experimentally demonstrated in 2004 [21]. In the real teleportation and entanglement swapping systems of continuous variables, the entanglements of a two-mode squeezed state used as an EPR source are always imperfect, because perfect entanglement requires perfect squeezing, which corresponds to a infinite amount of energy of the light field, that is impossible to obtain in experiments. Therefore, criteria for evaluating the efficiency of realistic—and thus imperfect—teleportation of quantum states are necessary. The fidelity is defined as the criterion to characterize the quality of the teleportation of a quantum state. The fidelities of teleportation for a coherent state using a two-mode squeezed vacuum state and the non-local correlations over large distance in a entanglement swapping system have been deduced [19, 20]. The fidelities for transfer of nonclassical features in quantum teleportation of nonclassical states, such as the squeezed vacuum and Fock state, have also been discussed [22, 23]. It has been proved that the properties of each of the two modes in an entangled two-mode squeezed state are precisely those of that of a single-mode thermal state [24]. Actually, the entanglement swapping is to teleport one beam of the entangled two-mode squeezed state from one location to another where the other EPR beam is located; correlation between the two different EPR beams is created. The swapped correlation degree strongly depends on the fidelity of the teleportation of one of the EPR beams. Thus the entanglement swapping is related to the teleportation of single-mode thermal states. Unlike the coherent and squeezed states, both the input and output states for the single-mode thermal state teleportation are mixed states.

In this paper, we discuss the teleportation of a single-mode thermal state of light field from the viewpoint of quantum mechanical theory in the Schrödinger picture. The statistical distance between the teleported input and output single-mode thermal states is introduced for estimating the teleportation efficiency. The criterion of the statistical distance (SD) with regard to the quantum–classical boundary is also calculated.

This paper is organized as follows. A general theory of the state evolution of teleportation is described [25] in section 2. The calculations of the SD and the classical boundary are given in section 3. Section 4 is a brief conclusion.

2. The state evolution of teleportation for continuous variables

For teleportation of continuous variables, usually the two-mode squeezed vacuum state serves as the EPR entangled state. In the Fock state basis the density operator of the squeezed state has the form

$$\hat{\rho}_{1,2} = (1-\lambda^2)\sum_n\sum_{n'}(-\lambda)^{n+n'}|n,n\rangle_{1,2}\langle n',n'|, \tag{1}$$

where $\lambda = \tanh r$, r is the squeezing parameter and the subscripts 1, 2 denote the two coupled modes in the squeezed state respectively.

In the Glauber–Sudarshan coherent-state representation, the input single-mode thermal state is expressed by

$$\hat{\rho}_{\rm in} = \int d^2\alpha\, P(\alpha)|\alpha\rangle_{\rm in}\langle\alpha|, \tag{2}$$

where $|\alpha\rangle_{\rm in}$ represents a coherent state basis for input state, α is a complex variable and $P(\alpha)$ is called the p function, which can be viewed as a probability density function in the phase space, and is normalized

$$\int d^2\alpha\, P(\alpha) = 1. \tag{3}$$

The density operator of the total initial state consisting of the input unknown state and the EPR beams is

$$\hat{\rho}_0 = \hat{\rho}_{\rm in}\otimes\hat{\rho}_{1,2}. \tag{4}$$

To perform a joint measurement on the input teleported state and half of the EPR beams, the input state is combined with half of the EPR beams on a 50:50 beamsplitter and then the quadrature amplitude of one output mode from the beamsplitter and the quadrature phase of the other output mode are homodyne detected [5].

A lossless 50:50 beamsplitter which mixes the teleported state of the light field and one half of the EPR beams is described by the unitary operator $\hat{U}$ [26]

$$\hat{U} = \exp\left(\frac{\pi}{4}(\hat{a}_{\rm in}\hat{a}_1^+ - \hat{a}_1\hat{a}_{\rm in}^+)\right). \tag{5}$$

For simplicity, we have ignored the phase-shifts induced by the beamsplitter in equation (5).

In the Schrödinger picture, the density matrix of the output state of the beamsplitter is obtained from the density matrix $\hat{\rho}_0$ of the initial state by the unitary transformation [27]

$$\hat{\rho}_{\rm BS} = \hat{U}^+\hat{\rho}_0\hat{U}. \tag{6}$$

Using the decomposition formula of the $SU(2)$ Lie algebra [28],

$$\hat{U}^+ = e^{\hat{a}_{\rm in}^+\hat{a}_1}(\sqrt{2})^{\hat{a}_{\rm in}^+\hat{a}_{\rm in}-\hat{a}_1^+\hat{a}_1}e^{-\hat{a}_{\rm in}\hat{a}_1^+}, \tag{7}$$

then substituting equations (1), (2) and (7) into equation (6), the density matrix of the output state of the beamsplitter $\hat{\rho}_{\rm BS}$ is expanded in the coherent state basis,

$$\begin{aligned}\hat{\rho}_{\rm BS} = {}& \frac{(1-\lambda^2)}{\pi^4}\int d^2\alpha\, P(\alpha)\\ &\otimes\int\int d^2\beta\, d^2\gamma\, e^{-\frac{3}{4}|\gamma|^2}e^{\frac{1}{2}|\lambda\gamma|^2}e^{-\frac{1}{2}|\beta|^2}e^{-\frac{1}{2}|\alpha|^2}e^{-\alpha\gamma^*}\\ &\times e^{\beta^*\gamma/\sqrt{2}}e^{\sqrt{2}\alpha\beta^*}|\beta\rangle_{\rm in}|\gamma/\sqrt{2}\rangle_1|-\lambda\gamma^*\rangle_2\\ &\otimes\int\int d^2\zeta\, d^2\eta_2\,\langle-\lambda\eta^*|_1\langle\eta/\sqrt{2}|_{\rm in}\langle\zeta|e^{-\frac{3}{4}|\eta|^2}e^{\frac{1}{2}|\lambda\eta|^2}\\ &\times e^{-\frac{1}{2}|\zeta|^2}e^{-\frac{1}{2}|\alpha|^2}e^{-\alpha^*\eta}e^{\zeta\eta^*/\sqrt{2}}e^{\sqrt{2}\alpha^*\zeta},\end{aligned} \tag{8}$$

where we have used the following relations:

$$\begin{aligned}(\sqrt{2})^{-\hat{a}_1^+\hat{a}_1}|\gamma\rangle_1 &= e^{-\frac{1}{4}|\gamma|^2}|\gamma/\sqrt{2}\rangle_1,\\ (\sqrt{2})^{\hat{a}_{\rm in}^+\hat{a}_{\rm in}}|\alpha\rangle_{\rm in} &= e^{\frac{1}{2}|\alpha|^2}|\sqrt{2}\alpha\rangle_{\rm in}.\end{aligned} \tag{9}$$

The amplitude quadrature $\hat{X}$ of one output beam of the beamsplitter and the phase quadrature $\hat{Y}$ of the other one are simultaneously measured by two homodyne detection systems. In the Schrödinger picture the measured outcomes correspond to the amplitude quadrature of the input state and the phase quadrature of half of the EPR beams respectively:

$$\begin{aligned}\hat{X} &= (\hat{a}_{\rm in}+\hat{a}_{\rm in}^+)/2,\\ \hat{Y} &= -i(\hat{a}_1-\hat{a}_1^+)/2.\end{aligned} \tag{10}$$

According to the general theory of quantum mechanical measurement, the quantum measurement is mathematically described by a positive operator-valued measurement (POVM) including a projection operator [29, 30]. The positive operator-valued measurement describing the homodyne detection of the two quadrature phases X and Y is given by

$$\begin{aligned}\prod_{\rm in}(X) &= |X\rangle_{\rm in}\langle X|,\\ \prod_1(Y) &= |Y\rangle_1\langle Y|,\end{aligned} \tag{11}$$

where X and Y are the outcomes of measurements. The states $|X\rangle_{\rm in}$ and $|Y\rangle_1$ are the eigenstates of the quadrature components. They satisfy the completeness relations: $\int_{-\infty}^{\infty}|X\rangle_{\rm in}\langle X|dX = \hat{I}$, $\int_{-\infty}^{\infty}|Y\rangle_1\langle Y|dY = \hat{I}$. During the process of measurement the other half of the EPR beams collapses due to the nonlocal entanglement and the normalized density matrix of the conditional output state becomes

$$\hat{\rho}_2(X,Y) = \frac{{\rm Tr}_{\rm in,1}\{\hat{\rho}_{\rm BS}\prod_{\rm in}(X)\prod_1(Y)\}}{P(X,Y)}, \tag{12}$$

where ${\rm Tr}_{\rm in,1}$ stands for the trace operation with respect to the input state and the first half of the EPR beams. $P(X,Y)$ is the probability distribution of the measured results:

$$P(X,Y) = {\rm Tr}_2\,{\rm Tr}_{\rm in,1}\left\{\hat{\rho}_{\rm BS}\prod_{\rm in}(X)\prod_1(Y)\right\}, \tag{13}$$

where ${\rm Tr}_2$ stands for the trace operation with respect to the other half of the EPR beams.

The eigenstates of the quadrature components in the coherent state representation are used in calculating the conditional output state of the second half of the EPR beams, that are

$$
\begin{aligned}
{}_{\rm in}\langle X|\beta\rangle_{\rm in} &= \left(\frac{2}{\pi}\right)^{1/4} \mathrm{e}^{-X^2+2\beta X-\frac{1}{2}|\beta|^2-\frac{1}{2}\beta^2} \\
{}_1\langle Y|\gamma/\sqrt{2}\rangle_1 &= \left(\frac{2}{\pi}\right)^{1/4} \mathrm{e}^{-Y^2+\mathrm{i}\sqrt{2}\gamma Y-\frac{1}{4}|\gamma|^2+\frac{1}{4}\gamma^2}.
\end{aligned} \tag{14}
$$

Substituting equations (11) and (14) into (12) and integrating out the parameters β and γ, the equation (12) becomes

$$
\hat{\rho}_2(X,Y) = \frac{(2/\pi)(1-\lambda^2)\mathrm{e}^{-2(X^2+Y^2)}\int \mathrm{d}^2\alpha\, P(\alpha)|\Psi\rangle\langle\Psi|}{P(X,Y)}, \tag{15}
$$

where $|\Psi\rangle = f(\alpha)\hat{D}(\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)])|0\rangle_2$, and $f(\alpha) = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\sqrt{2}\alpha(X+\mathrm{i}Y)}\mathrm{e}^{\frac{1}{2}\lambda^2|\alpha-\sqrt{2}(X-\mathrm{i}Y)|^2}$; the displacement operator is given by

$$
\begin{aligned}
\hat{D}(\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)]) &= \mathrm{e}^{-\frac{1}{2}\lambda^2|\alpha-\sqrt{2}(X-\mathrm{i}Y)|^2} \\
&\quad\times \mathrm{e}^{\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)]\hat{a}_2^+}\mathrm{e}^{\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)]\hat{a}_2}.
\end{aligned}
$$

Then equation (15) becomes

$$
\begin{aligned}
\hat{\rho}_2(X,Y) &= \frac{(2/\pi)(1-\lambda^2)}{P(X,Y)}\mathrm{e}^{-2(X^2+Y^2)}\int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha) \\
&\quad\times \hat{D}(\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)])|0\rangle_2\langle 0| \\
&\quad\times \hat{D}^+(\lambda[\alpha-\sqrt{2}(X-\mathrm{i}Y)]).
\end{aligned} \tag{16}
$$

Using the above obtained equations, the probability distribution equation (13) is simplified to

$$
P(X,Y) = (2/\pi)(1-\lambda^2)\mathrm{e}^{-2(X^2+Y^2)}\int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha). \tag{17}
$$

We note that, for $\lambda = 1$, i.e., for the ideal entangled EPR beams, the probability of the measured observables goes to zero. This implies that no information on the input state can be extracted from the classically measured outcomes, that is just the necessary requirement for faithful teleportation.

The last step of teleportation is to perform a unitary transformation on the other half of the EPR beams with the results measured at Alice's sending station to retrieve the input unknown state. For the case of continuous variables the unitary transformation is a displacement transformation $\hat{D}(Z)$; here $Z \propto X - \mathrm{i}Y$ is associated with the classical information of the infinitesimal measured values X and Y [31]. Then equation (16) is transformed into

$$
\begin{aligned}
\hat{\rho}_2^{\rm out} &= \frac{(2/\pi)(1-\lambda^2)}{P(X,Y)}\mathrm{e}^{-2(X^2+Y^2)}\int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha) \\
&\quad\times \hat{D}(\lambda\alpha+[Z-\sqrt{2}\lambda(X-\mathrm{i}Y)])|0\rangle_2\langle 0| \\
&\quad\times \hat{D}^+(\lambda\alpha+[Z-\sqrt{2}\lambda(X-\mathrm{i}Y)]).
\end{aligned} \tag{18}
$$

For a system of teleportation, we take $Z = \sqrt{2}g(X-\mathrm{i}Y)$ and equation (18) becomes

$$
\begin{aligned}
\hat{\rho}_2^{\rm out} &= \frac{(2/\pi)(1-\lambda^2)}{P(X,Y)}\mathrm{e}^{-2(X^2+Y^2)}\int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha) \\
&\quad\times \hat{D}(\lambda\alpha+\sqrt{2}(g-\lambda)(X-\mathrm{i}Y))|0\rangle_2\langle 0| \\
&\quad\times \hat{D}^+(\lambda\alpha+\sqrt{2}(g-\lambda)(X-\mathrm{i}Y)),
\end{aligned} \tag{19}
$$

where g represents the normalized classical gain for the transformation from classical measured values X and Y to complex field amplitudes Z. Equation (19) shows that the output state depends on the particularly measured results X and Y.

Considering the ideal condition of infinite squeezing with unit classical transformation gain, $\lambda = 1$, $g = 1$, we can obtain the density matrix of the conditional output state of the other half of the EPR pair:

$$
\begin{aligned}
\hat{\rho}_2^{\rm out} &= \frac{(2/\pi)(1-\lambda^2)\int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle_2\langle\alpha|}{P(X,Y)} \\
&= \int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle_2\langle\alpha| = \hat{\rho}_{\rm in}.
\end{aligned} \tag{20}
$$

Equation (20) shows that any unknown input quantum state can be perfectly reconstructed at the receiving station by means of the help of EPR nonlocal correlation under ideal conditions.

In the real world it is impossible to produce a perfect EPR pair, so a two-mode squeezed vacuum state with finite squeezing is used as the EPR source for the continuous variable teleportation. Thus the input and output states in the continuous variable teleportation system are not exactly the same using the imperfect EPR entanglement, only partly overlapped. Usually the fidelity is used for evaluating the quality of an imperfect teleportation. However, for the teleportation of a single-mode thermal state, a mixed state, the calculation of fidelity is not straightforward; this is why there are no publications discussing it so far. In the following we will introduce a statistical distance to estimate the quantum efficiency of a thermal state teleportation.

3. Statistical distance between input and output states of single-mode thermal state teleportation

Now we consider a general and thus practical case of $\lambda < 1$ and $g < 1$, which corresponds to the non-ideal EPR entanglement and non-unit gain. For continuous variables, the output state behaves like a mixture of the un-normalized density matrix elements. The average density matrix of the output state is

$$
\begin{aligned}
\hat{\rho}_{\rm out} &= (2/\pi)(1-\lambda^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}X\,\mathrm{d}Y\,\mathrm{e}^{-2(X^2+Y^2)} \\
&\quad\times \int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha) \\
&\quad\times \hat{D}(\lambda\alpha+\sqrt{2}(g-\lambda)(X-\mathrm{i}Y))|0\rangle_2\langle 0| \\
&\quad\times \hat{D}^+(\lambda\alpha+\sqrt{2}(g-\lambda)(X-\mathrm{i}Y)).
\end{aligned} \tag{21}
$$

The calculated result satisfies the normalized relation for the total output light field:

$$
\begin{aligned}
P(X,Y) &= (2/\pi)(1-\lambda^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}X\,\mathrm{d}Y\,\mathrm{e}^{-2(X^2+Y^2)} \\
&\quad\times \int \mathrm{d}^2\alpha\, P(\alpha)f(\alpha)f^*(\alpha) = 1.
\end{aligned}
$$

To evaluate the efficiency of realistic—and thus imperfect—teleportation of quantum states, some measures are needed to indicate the similarity between the teleported state and the input state; one such measure is called the fidelity, which is successfully used to deal with the continuous variable teleportation of a pure quantum state [5]. But for quantum teleportation of mixed states, e.g. the teleportation of a single-mode thermal state in our discussion, it may not be easy to obtain the result with the so-called mixed-state fidelity defined as [32]

$$F = \{\mathrm{Tr}[(\sqrt{\hat{\rho}_{\mathrm{in}}}\hat{\rho}_{\mathrm{out}}\sqrt{\hat{\rho}_{\mathrm{in}}})^{1/2}]\}^2 \tag{22}$$

where $\hat{\rho}_{\mathrm{in}}$ and $\hat{\rho}_{\mathrm{out}}$ are the density operators of the input and output states respectively. The square root of the density operator appearing in the above equation introduces extra difficulties in calculating the fidelity of the mixed state.

Thus, instead of using the fidelity for the mixed state, we use the statistical distance (SD) between two density operators of input and output states to evaluate their difference. Based on the Hilbert–Schmidt norm [33], the SD is defined:

$$d = \left[\tfrac{1}{2}\,\mathrm{Tr}(\hat{\rho}_{\mathrm{in}})^2 + \tfrac{1}{2}\,\mathrm{Tr}(\hat{\rho}_{\mathrm{out}})^2 - \mathrm{Tr}(\hat{\rho}_{\mathrm{in}}\hat{\rho}_{\mathrm{out}})\right]^{1/2}; \tag{23}$$

obviously, the square root acting on the trace of the density operators in the SD leads to an easy calculation process. Unlike the definition of fidelity, for which $F = 1$ corresponds to a case of $\hat{\rho}_{\mathrm{out}} = \hat{\rho}_{\mathrm{in}}$, SD reflects the efficiency of the teleportation from the opposite side. If $\hat{\rho}_{\mathrm{out}}$ is exactly equal to $\hat{\rho}_{\mathrm{in}}$, we can get $d = 0$; on the contrary, if $d > 0$, and the bigger d is, the larger the distance between input and output states is.

For an input single-mode thermal state with the average photon numbers $\bar{n}$, we have [24]

$$P(\alpha) = \frac{1}{\pi\bar{n}}\mathrm{e}^{-\frac{|\alpha|^2}{\bar{n}}}, \tag{24}$$

$$\hat{\rho}_{\mathrm{in}} = (1 - \mathrm{e}^{-\mu})\mathrm{e}^{-\mu\hat{a}^+\hat{a}}, \tag{25}$$

where $\mathrm{e}^{-\mu} = 1/(\frac{1}{\bar{n}} + 1)$.

Substituting equations (25), (24) and (21) into equation (23), we obtain

$$D = [A + B - C]^{1/2}, \tag{26}$$

where

$$A = \frac{1}{2(2\bar{n}+1)}$$

$$B = \left\{\frac{1}{2[(1-\lambda^4)\bar{n}^2 + 2\bar{n} + 1]}\right\}\frac{1}{EF}$$

$$C = \left\{\frac{1}{(1-\lambda^2)\bar{n}^2 + 2\bar{n} + 1}\right\}\frac{1}{H}$$

$$\begin{aligned}
E = 1 &+ \frac{(g-\lambda)^2}{1-\lambda^2} - \frac{(1-g\lambda)^2}{1-\lambda^2}\frac{\bar{n}}{\bar{n}+1} \\
&- \frac{\lambda^2(g-\lambda)^2}{1-\lambda^2}\frac{\bar{n}(\bar{n}+1)}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1} \\
&- \frac{\lambda^4(1-g\lambda)^2}{1-\lambda^2}\frac{\bar{n}^3}{[(1-\lambda^4)\bar{n}^2+2\bar{n}+1](\bar{n}+1)} \\
&- \frac{2\lambda^3(1-g\lambda)(g-\lambda)}{1-\lambda^2}\frac{\bar{n}^2}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1}
\end{aligned}$$

$$\begin{aligned}
F = 1 &+ \frac{(g-\lambda)^2}{1-\lambda^2} - \frac{\lambda^2(g-\lambda)^2}{1-\lambda^2}\frac{\bar{n}}{\bar{n}+1} \\
&- \frac{(1-g\lambda)^2}{1-\lambda^2}\frac{\bar{n}(\bar{n}+1)}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1} \\
&- \frac{\lambda^6(g-\lambda)^2}{1-\lambda^2}\frac{\bar{n}^3}{[(1-\lambda^4)\bar{n}^2+2\bar{n}+1](\bar{n}+1)} \\
&- \frac{2\lambda^3(1-g\lambda)(g-\lambda)}{1-\lambda^2}\frac{\bar{n}^2}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1} - \frac{G^2}{E}
\end{aligned}$$

$$\begin{aligned}
G = &\frac{(g-\lambda)^2}{1-\lambda^2} + \frac{\lambda(g-\lambda)(1-g\lambda)}{1-\lambda^2}\frac{\bar{n}}{\bar{n}+1} \\
&- \frac{\lambda(g-\lambda)(1-g\lambda)}{1-\lambda^2}\frac{\bar{n}(\bar{n}+1)}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1} \\
&- \frac{\lambda^5(g-\lambda)(1-g\lambda)}{1-\lambda^2}\frac{\bar{n}^3}{[(1-\lambda^4)\bar{n}^2+2\bar{n}+1](\bar{n}+1)} \\
&- \frac{\lambda^2(1-g\lambda)^2+\lambda^4(g-\lambda)^2}{1-\lambda^2}\frac{\bar{n}^2}{(1-\lambda^4)\bar{n}^2+2\bar{n}+1}
\end{aligned}$$

$$\begin{aligned}
H = 1 &+ \frac{(g-\lambda)^2}{1-\lambda^2}\frac{1}{\bar{n}+1} - \frac{[(1-\lambda^2) - \frac{\lambda(g-\lambda)}{\bar{n}+1}]^2}{1-\lambda^2} \\
&\times \frac{\bar{n}(\bar{n}+1)}{(1-\lambda^2)\bar{n}^2+2\bar{n}+1}.
\end{aligned}$$

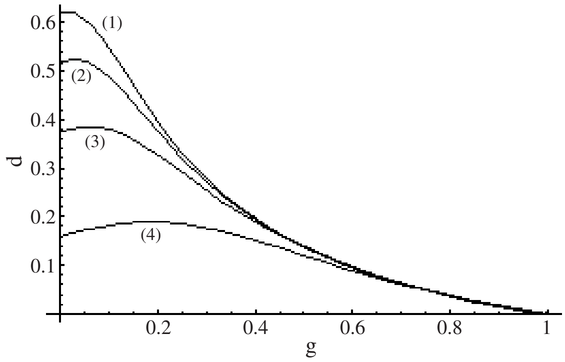

Figure 1. The dependences of SD on the normalized classical gain g and EPR entanglement λ are plotted for the mean photon numbers $\bar{n} = 20$. (1) $\lambda = 0.3$; (2) $\lambda = 0.5$; (3) $\lambda = 0.7$; (4) $\lambda = 0.9$.

When taking $\lambda = 1$, $g = 1$, equation (26) gives $d = 0$; this demonstrates $\hat{\rho}_{\mathrm{out}} = \hat{\rho}_{\mathrm{in}}$, that is the output state perfectly mimics the input state.

The curves in figure 1 show the dependence of the SD on the normalized classical gain (g) and EPR entanglement (λ) at given mean photon numbers of thermal state $\bar{n} = 20$. The entanglements of curves (1)–(4) are respectively 0.3, 0.5, 0.7 and 0.9. This shows, for getting high teleportation efficiency, i.e. small SD, large classical gain and high EPR entanglement have to be required. The dependence of the SD on mean photon numbers and normalized gain for a given EPR entanglement ($\lambda = 0.7$) is plotted in figure 2. We can see that for a thermal state with larger photon numbers, the teleportation of higher efficiency with lower SD can be achieved with a gain lower than that needed for a thermal state with low mean photon numbers.

The boundary between the classical and quantum teleportation of single-mode thermal states is created by setting $\lambda = 0$ and $g = 1$,

$$d_{\mathrm{boundary}} = \left[\frac{1}{2(2\bar{n}+1)} + \frac{1}{2(2\bar{n}+3)} - \frac{1}{2\bar{n}+2}\right]^{1/2}. \tag{27}$$

It is seen that the boundary is strongly dependent on the average photon number of thermal states; for input thermal

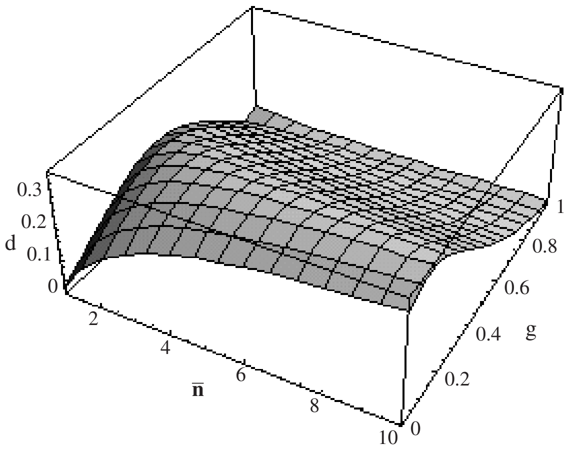

Figure 2. Plot of the SD between the input and output states as a function of mean photon number $\bar{n}$ and the normalized classical gain g. Here the EPR entanglement is given, $\lambda = 0.7$.

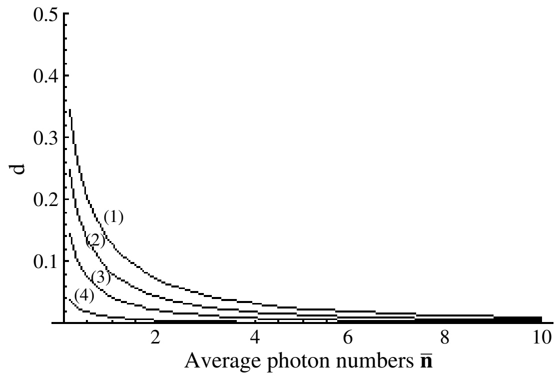

Figure 3. The dependences of the SD between the teleported input and output states on the mean photon number $\bar{n}$ of a single-mode thermal field in the case of unit classical gain $g = 1$. $\lambda = 0$ for curve (1) is the classical–quantum boundary; curves (2), (3) and (4), $\lambda = 0.3$, $\lambda = 0.6$ and $\lambda = 0.9$ respectively.

states with different average numbers, the boundary justifying quantum teleportation should have a different criterion; when $d < d_{\mathrm{boundary}}$ the teleportation in the quantum domain is reached.

The dependences of the SD on the photon numbers and the degree of entanglement are plotted in figure 3 for unit classical gain $g = 1$. Curve (1) stands for the boundary. Curves (2), (3) and (4) correspond to $\lambda = 0.3$, 0.6 and 0.9. For a given EPR entanglement, the larger the photon numbers are, the lower the SD is.

4. Conclusion

In conclusion, the quantum teleportation of single-mode thermal states has been theoretically analysed from the viewpoint of the general quantum mechanical theory of measurements. The statistical distance characterizing the efficiency of teleportation is deduced from the definition of the Hilbert–Schmidt norm. The dependences of the SD upon the entanglement λ, the mean photon number $\bar{n}$ of the thermal state and the gain g of the system are numerically calculated, and the classical–quantum boundary for the SD is established for the first time to the best of our knowledge.

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